

A NOTE ON MONOTONICITY OF MIXED RAMSEY NUMBERS

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ABSTRACT. For two graphs, G , and H , an edge-coloring of a complete graph is (G, H) -good if there is no monochromatic subgraph isomorphic to G and no rainbow subgraph isomorphic to H in this coloring. The set of number of colors used by some (G, H) -colorings of K_n is called a mixed-Ramsey spectrum. This note addresses a fundamental question of whether the spectrum is an interval. It is shown that the answer is “yes” if G is not a star and H does not contain a pendent edge.

1. INTRODUCTION

Let G and H be two graphs on fixed number of vertices. An edge coloring of a complete graph, K_n , is called (G, H) -good if there is no monochromatic copy of G and no rainbow (totally multicolored) copy of H in this coloring. This, sometimes called mixed-Ramsey coloring, is a hybrid of classical Ramsey and anti-Ramsey colorings, [18, 6]. As shown by Jamison and West [15], a (G, H) -good coloring of an arbitrarily large complete graph exists unless either G is a star or H is a forest.

Let $S(n; G, H)$ be the set of the number of colors, k , such that there is a (G, H) -good coloring of K_n with k colors. We call $S(n; G, H)$ a spectrum. Let $\max S(n; G, H)$, $\min S(n; G, H)$ be the maximum, minimum number in $S(n; G, H)$, respectively. The behavior of these functions was studied in [2], [8], [1] and others. Note that if there is no restriction on a graph H , $S(n; G, *)$ is an interval $[k, \binom{n}{2}]$, where k is the largest number such that $r_{k-1}(G) \leq n$, a classical multicolor Ramsey number.

The main question investigated in this note is whether the same behavior continues to hold for mixed Ramsey colorings. Specifically, for given integer n and graphs G and H , is $S(n; G, H)$ an interval? When G is not a star, for most graphs H , we show that $S(n; G, H)$ is an interval.

Theorem 1. *Let G be a graph that is not a star, and let H be a graph with minimum degree at least 2. Then for any natural number n , $S(n; G, H)$ is an interval.*

The simplest connected graph H which is not a tree and which has a vertex of degree 1 is $K_3 + e$, a 4-vertex graph obtained by attaching a pendent edge to a triangle. We show that $S(n; G, K_3 + e)$ could have a gap for some graphs G and some values of n . However, when n is arbitrarily large, we do not have a single example of a graph G and a graph H for which $S(n; G, H)$ is not an interval.

Specifically, the next theorem is a collection of results on $S(n; G, K_3 + e)$. Here, ℓK_2 is a matching of size ℓ , C_4 is a 4-cycle, and P_4 is a path on 4 vertices.

Theorem 2.

- $S(n; \ell K_2, K_3) = S(n; \ell K_2, K_3 + e) = [\lceil \frac{n-2\ell+1}{\ell-1} \rceil + 1, n-1]$, $n \geq 4$,
 $S(n; P_4, K_3) = S(n; P_4, K_3 + e) = [n-2, n-1]$, $n \geq 4$,
 $S(n; C_4, K_3) = S(n; C_4, K_3 + e) = [n-3, n-1]$, $n \geq r_3(C_4) = 11$,
 $S(n; K_3, K_3) = S(n; K_3, K_3 + e) = [c \log n, n-1]$, $n \geq r_3(K_3) = 17$,
 $S(n; K_{1,\ell}, K_3) = S(n; K_{1,\ell}, K_3 + e) = \emptyset$, $n \geq 3\ell + 1$.

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- $S(10; C_4, K_3 + e) = \{3, 7, 8, 9\}$.

Corollary 3. *If $\ell \geq 2$ and $n \geq \max\{17, 3\ell + 1\}$, then $S(n; G, K_3 + e)$ is an interval for any $G \in \{K_3, \ell K_2, C_4, P_4, K_{1,\ell}\}$. However, $S(n; G, K_3 + e)$ is not an interval if $n = 10$ and $G = C_4$.*

Open question. Are there graphs G and H such that for any natural number N there is $n > N$ so that $S(n; G, H)$ is not an interval?

2. DEFINITIONS AND PROOFS OF MAIN RESULTS

For an edge coloring c of K_n and a vertex $x \in V(K_n)$, let $N_c(x)$ be the set of colors used only on edges incident to x , and for $X \subseteq V(K_n)$ let $c(X)$ be the set of colors used on edges induced by X . Let $|c|$ denote the number of colors used in the coloring c . Then $|c| = |N_c(x)| + |c(V \setminus x)|$ for any $x \in V$. We shall use function

$$f(k; G, H) = \max\{n : \text{there is a } (G, H)\text{-good coloring of } K_n \text{ using exactly } k \text{ colors}\}.$$

Note that if $f(k; G, H) = n$, then $\min S(n; G, H) = k$.

Observation 1 If G is not a star, and A and B are color classes which are stars with the same center in a (G, H) -good coloring c of K_n with k colors, then replacing A and B in c with a new color class $A \cup B$ gives a (G, H) -good coloring using $k - 1$ colors.

Observation 2 For any graphs G and H ,

$$\min S(n; G, H) \leq \min S(n + 1, G, H).$$

Proof. Consider a (G, H) -good coloring of K_{n+1} with k colors. Delete one vertex to get a (G, H) -good coloring of K_n with $k' \leq k$ colors. \square

Observation 3 For $G \subseteq G'$ and $H \subseteq H'$,

$$S(n; G, H) \subseteq S(n; G', H) \subseteq S(n; G', H') \quad \text{and} \quad S(n; G, H) \subseteq S(n; G, H') \subseteq S(n; G', H').$$

Proof. If there is no monochromatic G and no rainbow H in a coloring of $E(K_n)$, then there is no monochromatic G' and no rainbow H' in this coloring. \square

Observation 4 If G is not a star, H has minimum degree at least 2, and $k \in S(n; G, H)$, then $k + 1 \in S(n + 1; G, H)$.

Proof. Consider a (G, H) -good coloring of K_n with k colors. Add a new vertex x , and color edges incident to x by a new color to get a (G, H) -good coloring of K_{n+1} with $k + 1$ colors. \square

Proof of Theorem 1.

We need to prove that $[\min S(n; G, H), \max S(n; G, H)] \subseteq S(n; G, H)$. We use induction on n . When $n = 2$, any coloring uses one color. Let $n \geq 3$. Consider the smallest k such that $[k, \max S(n; G, H)] \subseteq S(n; G, H)$. Observe that in any (G, H) -good k -coloring of K_n and any vertex x , we have $|N(x)| \leq 1$, otherwise applying Observation 1 gives us a (G, H) -good $(k - 1)$ -coloring of K_n violating minimality of k . Consider a (G, H) -good k -coloring of K_n and any vertex x , and delete it. Then we have a (G, H) -good coloring of K_{n-1} with k or $k -$

1 colors. Here we note that $\max S(n-1; G, H) \geq k-1$. By induction, $S(n-1; G, H)$ is an interval, i.e., $[\min S(n-1; G, H), \max S(n-1; G, H)] = S(n-1; G, H)$. Then by Observation 4, $[\min S(n-1; G, H) + 1, \max S(n-1; G, H) + 1] \subseteq S(n; G, H)$. Since $\min S(n; G, H) \geq \min S(n-1; G, H)$ from Observation 2, $[\min S(n; G, H), \max S(n-1; G, H) + 1] \subseteq S(n; G, H)$. Since $k \leq \max S(n-1; G, H) + 1$ and $[k, \max S(n; G, H)] \subseteq S(n; G, H)$ we finally have that $[\min S(n; G, H), \max S(n; G, H)] \subseteq S(n; G, H)$. \square

Proof of Theorem 2.

First observe that $\max S(n; G, H) \leq AR(n, H)$, where $AR(n, H)$ is the classical anti-Ramsey number, the maximum number of colors in an edge-coloring of K_n with no rainbow subgraphs isomorphic to H . If G is not a star, $\max S(n; G, K_3) = AR(n, K_3) = n-1$, see [2]. Moreover, from Observation 3, we obtain that $\max S(n; G, K_3) \leq \max S(n; G, K_3 + e)$; and from [12], we know that $AR(n, K_3) = AR(n, K_3 + e)$. Thus, when G is not a star, $\max S(n; G, K_3) = \max S(n; G, K_3 + e) = n-1$ for $n \geq 4$.

Therefore if $\min S(n; G, K_3) = \min S(n; G, K_3 + e)$, and G is not a star, we can conclude that $S(n; G, K_3 + e) = S(n; G, K_3)$, which is an interval by Theorem 1. Next, we shall analyze $\min S(n; G, K_3 + e)$. Recall that $\min S(n; G, H) = k$ if $f(k, G, H) = n$. Moreover, $f(k, G, H) + 1 \leq r_k(G)$, where $r_k(G)$ denotes the classical k -color Ramsey number for G . The equality holds if there is a k -coloring of $E(K_{r_k(G)-1})$ with no monochromatic G and no rainbow H .

Case 1. $G = \ell K_2$

From [17], we have that $r_k(\ell K_2) = (k-1)(\ell-1) + 2\ell$. The extremal coloring providing this Ramsey number can be constructed as follows. Consider a complete graph on $2\ell-1$ vertices colored entirely with color 1, add $\ell-1$ vertices and color all edges incident to these vertices with color 2, then add another $\ell-1$ vertices and color all edges incident to these vertices with color 3. Repeat this process until we get a k -coloring of a complete graph on $2\ell-1 + (k-1)(\ell-1)$ vertices which contains no monochromatic ℓK_2 . Note that this coloring contains no rainbow cycles, thus, it contains neither rainbow copy of K_3 nor rainbow copy of $K_3 + e$. Hence $\min S(n; \ell K_2, H) = \min S(n; \ell K_2, H + e)$ for any H , not a forest. In particular for $\ell \geq 2$, $\min S(n; \ell K_2, K_3) = \min S(n; \ell K_2, K_3 + e) = \lceil \frac{n-2\ell+1}{\ell-1} \rceil + 1$.

Case 2. $G \in \{K_3, P_4, C_4\}$

From [5, 2, 13, 7, 8] we have that $f(k, K_3, K_3) = f(k, K_3, K_3 + e) = \lambda(k)$, for $k \geq 4$, where $\lambda(k) = 5^{k/2}$ if k is even, $2 \cdot 5^{(k-1)/2}$ if k is odd; $f(k, P_4, K_3) = f(k, P_4, K_3 + e) = k+2$ for $k \geq 1$, and $f(k, C_4, K_3) = f(k, C_4, K_3 + e) = k+3$ for $k \geq 4$. Therefore $\min S(n; P_4, K_3) = \min S(n; P_4, K_3 + e) = n-2$, $\min S(n; C_4, K_3) = \min S(n; C_4, K_3 + e) = n-3$, and $\min S(n; K_3, K_3) = \min S(n; K_3, K_3 + e) = c \log n$. Thus $\min S(n; G, K_3) = \min S(n; G, K_3 + e)$ for $G \in \{K_3, P_4, C_4\}$ and $n \geq r_3(G)$.

Case 3. $G = K_{1,\ell}$

In [14], it was shown that any coloring of $E(K_n)$ with no rainbow triangles has a monochromatic star $K_{1,2n/5}$. Using this fact and the pigeonhole principle, we easily see that any coloring of $E(K_n)$ with no rainbow $K_3 + e$ has a monochromatic star $K_{1,n/3}$. This is sharp as is seen in [8]. Therefore $S(n; K_{1,\ell}, K_3) = S(n; K_{1,\ell}, K_3 + e) = \emptyset$ if $n > 3\ell$.

Summarizing 1), 2), and 3) we have that $S(n; G, K_3) = S(n; G, K_3 + e)$ is an interval if G is one of $\{\ell K_2, K_3, P_4, C_4, K_{1,\ell}\}$ and $n \geq N$, where N is a constant depending only on G . This concludes the proof of the first part of the Theorem.

Consider the case when $G = C_4$, $H = K_3 + e$ and $n = 10$. Since $r_2(C_4) = 6 < 10$, we see that there is no $(C_4, K_3 + e)$ -good coloring of K_{10} in two colors. On the other hand, since $r_3(C_4) = 11$,

there is a $(C_4, K_3 + e)$ -good coloring of K_{10} in three colors. Thus $\min S(10; C_4, K_3 + e) = 3$. We also have that $\max S(10; C_4, K_3 + e) = AR(10, K_3) = 9$. Since $f(k, C_4, K_3 + e) = k + 3 < 10$ for $4 \leq k \leq 6$, there is no $(C_4, K_3 + e)$ -good coloring of K_{10} with 4, 5, or 6 colors. To construct 8- and 7-colorings of K_{10} with no rainbow $K_3 + e$ and no monochromatic C_4 , consider a vertex set $\{v_1, \dots, v_{10}\}$. Let $c(v_i v_j) = i$, $1 \leq i \leq 7$, $i < j$; $c(v_8 v_9) = c(v_8 v_{10}) = c(v_9 v_{10}) = 8$. Let $c'(v_i v_j) = i$, $1 \leq i \leq 5$, $i < j$; $c'(v_6 v_7) = c'(v_7 v_8) = c'(v_8 v_9) = c'(v_9 v_{10}) = c'(v_{10} v_6) = 6$, all other edges get color 7 under c' . Note that c and c' are 8- and 7-colorings, respectively, containing no rainbow K_3 and no monochromatic C_4 . Thus $S(10; C_4, K_3 + e) = \{3, 7, 8, 9\}$. □

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